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Photorefractive accelerating pulses

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Abstract
The existence of similarity reductions having $x - 2Vz - \kappa z^2$ as similarity variable is demonstrated for a number of evolution systems arising in optoelectronics. The parameter κ ($\neq 0$) corresponds to acceleration if z represents time, but corresponds to curvature if z is the propagation distance. This is of current relevance to photorefractivity, which is known both theoretically and experimentally to predict self-guiding optical beams. The link between space-charge diffusion (which breaks the $x \leftrightarrow -x$ symmetry) and beam curvature is shown. A perturbation analysis for small diffusivity is used to predict how κ and the beam profile depend on beam power and on diffusivity. The results agree well with computations for the ordinary differential equations, which are continued to larger diffusivities over a wide range of beam powers.

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1. Introduction and motivating example

Nonlinear optoelectronics has motivated the study of numerous equations and systems which generalize the cubic nonlinear Schrödinger equation and which possess similarity reductions based on travelling wave coordinates, the diffusive similarity variable $xt^{-1/2}$ and combinations thereof. However, little attention has previously been given to the fact that many of these systems also possess reductions having as similarity variable the quantity $x - \kappa t^2$, which is constant along paths $\mathrm{d}x/\mathrm{d}t = 2\kappa t$ moving with constant acceleration. If the corresponding solutions are pulse-like, they are known as 'accelerating solitons' (although, strictly, they should be called accelerating solitary waves).

One of us first became aware of the accelerating similarity variable through a complete Lie group analysis (Parker 1988) of the coupled NLS equations

$$\begin{aligned} \mathrm{i}u_z + u_{xx} + (|u|^2 + \sigma|v|^2)u &= 0 \\ \mathrm{i}v_z + v_{xx} + (\sigma|u|^2 + |v|^2)v &= 0 \end{aligned} \qquad (1.1)$$

arising from the (symmetric) 'cross-phase modulation' of two modes in an optical waveguide. Here z is the propagation distance, x is a (scaled) distance through the pulse measured relative to an origin moving at the group velocity (common to each mode), while u and v are the complex amplitudes of each mode at the chosen carrier frequency and subscripts denote partial derivatives. The cubically nonlinear terms are not 'phase sensitive' and may be interpreted as depending on the intensities $|u|^2$ and $|v|^2$ in the two modes. The parameter σ is the ratio between 'cross-phase modulation' and 'self-phase modulation' coefficients; in fibres, it is positive and typically is close to 2.

Similarity analysis for the system (1.1) showed the existence of the following similarity variables:

$$\text{(i) } x - 2Vz, \quad \text{(ii) } z, \quad \text{(iii) } xz^{-1/2}, \quad \text{(iv) } (x - 2Vz)z^{-1/2}, \quad \text{(v) } x - \kappa z^2.$$

Similarity variable (i) corresponds to travelling wave solutions, the variable (ii) arises for solutions with intensity depending only on z, (iii) is familiar for the heat equation, (iv) is a combination of (i) and (iii), while variable (v) is the accelerating similarity variable of present interest. (An investigation (Manganaro and Parker 1993) of systems generalizing (1.1) and having variable coefficients used the Clarkson and Kruskal (1989) procedure, but found no additional similarity variables.)

For the system (1.1), the ordinary differential equations (ODEs) determining the structure of similarity solutions are autonomous in cases (i) and (ii) (i.e. the similarity variable does not appear *explicitly* in the coefficients), but are non-autonomous otherwise. In case (ii), they are readily solved. However, it is cases (i) and (v) which are relevant in this paper.

For case (i), the similarity reduction may be written as

$$u = \mathrm{e}^{\mathrm{i}\theta(z,\eta)} F(\eta) \qquad v = \mathrm{e}^{\mathrm{i}\phi(z,\eta)} G(\eta) \qquad \eta \equiv x - 2Vz$$

with F, G, θ and ϕ real. When these are inserted into (1.1), it is found that

$$\begin{aligned} F''(\eta) + \left\{2V\theta_\eta - (\theta_\eta)^2 - \theta_z\right\} F + (F^2 + \sigma G^2)F &= 0 \\ G''(\eta) + \left\{2V\phi_\eta - (\phi_\eta)^2 - \phi_z\right\} G + (\sigma F^2 + G^2)G &= 0 \end{aligned} \tag{1.2}$$

with two further equations which are equivalent to

$$\partial\left\{(\theta_\eta - V)F^2(\eta)\right\}/\partial\eta = 0 \qquad \partial\left\{(\phi_\eta - V)G^2(\eta)\right\}/\partial\eta = 0. \tag{1.3}$$

These may be integrated twice to give

$$\begin{aligned} \theta &= V\eta + B_1(z) + \int C_1(z)F^{-2}(\eta)\,\mathrm{d}\eta \\ \phi &= V\eta + B_2(z) + \int C_2(z)G^{-2}(\eta)\,\mathrm{d}\eta. \end{aligned} \tag{1.4}$$

Substitution into equations (1.2) is then consistent only if C_1 and C_2 are constants while $B_i(z) = \Gamma_i z$ $(i = 1, 2)$, hence yielding the structure ODEs

$$\begin{aligned} F''(\eta) + \left\{V^2 - \Gamma_1 - C_1^2F^{-4}(\eta) + F^2 + \sigma G^2\right\} F &= 0 \\ G''(\eta) + \left\{V^2 - \Gamma_2 - C_2^2G^{-4}(\eta) + \sigma F^2 + G^2\right\} G &= 0. \end{aligned} \tag{1.5}$$

Observe that equations (1.5) may have pulse-like solutions (i.e. $F \to 0, G \to 0$ as $\eta \to \pm\infty$) only if $C_1 = C_2 = 0$ (which is analogous to the vanishing of 'angular momentum' in many mechanical systems, see the appendix). The numerical evidence is that families possessing any specified number n of maxima for $|F|$ exist. Also, certain solutions of (1.5) are known explicitly (Porubov and Parker 1999) in terms of hyperbolic and Jacobian elliptic functions.

For case (v), with similarity variable $\eta = x - \kappa z^2$, the representations $u = e^{i\theta} F(\eta)$ and $v = e^{i\phi} G(\eta)$ are unchanged, while (1.2) and (1.3) are altered only by the replacement $V \to \kappa z$. Expressions (1.4) are replaced by

$$\begin{aligned} \theta &= \kappa z \left(\eta + \frac{1}{3} \kappa z^2 \right) + \Gamma_1 z + \int C_1 F^{-2} \, d\eta \\ \phi &= \kappa z \left(\eta + \frac{1}{3} \kappa z^2 \right) + \Gamma_2 z + \int C_2 G^{-2} \, d\eta \end{aligned} \tag{1.6}$$

while the structure ODEs (1.5) are replaced by

$$\begin{aligned} F''(\eta) + \left\{ -\kappa\eta - \Gamma_1 - C_1^2 F^{-4}(\eta) + F^2 + \sigma G^2 \right\} F &= 0 \\ G''(\eta) + \left\{ -\kappa\eta - \Gamma_2 - C_2^2 G^{-4}(\eta) + \sigma F^2 + G^2 \right\} G &= 0. \end{aligned} \tag{1.7}$$

Equations (1.7) are merely a variant of (1.5), in which the non-autonomous terms $-\kappa\eta$ arise due to the 'drift' in the 'frequencies' θ_η and ϕ_η which is proportional to z and is induced by the 'acceleration' κ. The close similarity of the systems (1.5) and (1.7) suggests that travelling waves and accelerating similarity reductions may usefully be analysed simultaneously, using the composite similarity variable

$$\eta = x - 2Vz - \kappa z^2. \tag{1.8}$$

In fact, for the system (1.1) (and indeed many others), pulse-like solutions can have similarity variable (1.8) only for $\kappa = 0$, so that there is no acceleration. However, some systems which are not invariant under the mapping $x \leftrightarrow -x$ are found to have accelerating pulse-like solutions. Examples are pulses in active media (Vanin *et al* 1994) and optical soliton transmission subject to sliding-frequency filters in the limit of small amplifier spacing (Kodama and Wabnitz 1994). These are discussed briefly in section 2, illustrating procedures to be developed in section 3 for photorefractive self-guiding beams, such as those investigated for coupled modes by Singh *et al* (1995). In section 4, the structure of localized (soliton) solutions is analysed for the single-mode case, showing that acceleration (strictly a 'bending' effect for a spatial beam) is absent unless diffusion of space charge is included (so breaking the $x \leftrightarrow -x$ symmetry). A relation between path curvature and the diffusion parameter is derived, involving integrated properties of the beam, such as power. To aid in numerical determination of beam profiles, an asymptotic description is developed in section 5. The resulting beam profiles are shown in section 6 for various values of the diffusion parameter, over a range of beam powers. It is shown, over a considerable range of powers, that the path curvature is well predicted by the asymptotic description, in which the beam profile is symmetric as determined in the diffusionless case. At all values of diffusion, there is a beam power at which the path bending is maximum; moreover, this power is largely insensitive to the diffusion parameter.

2. The 'accelerating' similarity reduction

The form of equations (1.4)–(1.7) suggests that many systems, including those of the form

$$i\partial_z u_n + D_n \partial_x^2 u_n + F_n \left(|u_1|^2, \ldots, |u_N|^2 \right) u_n = 0 \qquad n = 1, 2, \ldots, N \tag{2.1}$$

with coefficients D_n and functions F_n which are real, possess similarity reductions both based on the accelerating similarity variable $x - \kappa z^2$ and the familiar travelling wave variable $x - 2Vz$. Accordingly, we demonstrate that equations (2.1) are compatible with the composite similarity *ansatz*

$$u_n = e^{i\theta_n(z,\eta)} U_n(\eta) \qquad \eta = x - 2Vz - \kappa z^2 \tag{2.2}$$

with U_n and θ_n real (for $n = 1, 2, \ldots, N$). (Actually, Lie group analysis, following Bluman and Cole (1974) by seeking continuous groups of transformations which leave (2.1) invariant, shows that, except when F_n is homogeneous of degree 1 in its N arguments, as in coupled cubic NLS systems, expressions (2.2) describe all self-similar solutions other than certain non-propagating solutions of the form

$$u_n = \mathrm{e}^{\mathrm{i}(x+k_n)^2/4D_n z} v_n(z) \qquad \text{for which} \quad v_n = z^{-1/2} C_n \mathrm{e}^{\mathrm{i}r_n(z)}$$

with C_n and $r_n(z)$ real.)

Experience with many evolution equations indicates that self-similar solutions frequently describe the eventual asymptotic form of solutions corresponding to broad classes of initial conditions. Hence, solutions (2.2) with $U_n \to 0$ as $\eta \to \pm\infty$ are expected to describe propagation of localized pulses.

Since $\eta_z = -2(V + \kappa z)$ and $\eta_x = 1$, we find that equations (2.1) and (2.2) give rise to

$$D_n U_n'' + \left\{2(V + \kappa z)\partial_\eta \theta_n - D_n(\partial_\eta \theta_n)^2 - \partial_z \theta_n\right\} U_n + F_n\left(U_1^2, U_2^2, \ldots, U_N^2\right) U_n + \mathrm{i}\left\{2(D_n \partial_\eta \theta_n - V - \kappa z)U_n' + D_n\left(\partial_\eta^2 \theta_n\right) U_n\right\} = 0. \tag{2.3}$$

Multiplying the imaginary part by U_n and integrating readily yields the results

$$\partial_\eta \theta_n = \frac{V + \kappa z}{D_n} + \frac{C_n(z)}{U_n^2(\eta)} \qquad n = 1, 2, \ldots, N$$

which may be integrated again to give

$$\theta_n(z, \eta) = D_n^{-1}(V + \kappa z)\eta + \Theta_n(z) + \int C_n(z) U_n^{-2}\, \mathrm{d}\eta. \tag{2.4}$$

Since, in the real part of (2.3), the coefficient of U_n cannot depend on z, substitution from (2.4) shows that, for each n, both C_n and $\theta_n'(z) - (V + \kappa z)^2/D_n$ are constants. Consequently, generalizations of both (1.3) and (1.5) are obtained as

$$\arg u_n \equiv \theta_n(z, \eta) = D_n^{-1}\left\{(V + \kappa z)\eta + \tfrac{1}{3}\kappa^2 z^3 + V\kappa z^2\right\} + \Gamma_n z + \int C_n U_n^{-2}\, \mathrm{d}\eta. \tag{2.5}$$

The corresponding structure ODEs form the non-autonomous system

$$D_n U_n''(\eta) - \left\{\Gamma_n + D_n^{-1}(\kappa\eta - V^2) + D_n C_n^2 U_n^{-4}(\eta) - F_n\left(U_1^2, \ldots, U_N^2\right)\right\} U_n = 0. \tag{2.6}$$

Besides the systems (1.1) and (2.1), others possess the accelerating similarity reduction. Until 1994, accelerating solutions seemed to be merely a mathematical curiosity, but then Vanin *et al* (1994) treated ultra-short laser-generated pulses travelling through an active medium by studying the single equation

$$-\mathrm{i}A_z + A_{xx} + |A|^2 A + \mathrm{i}A\left\{1 - \int_{-\infty}^{x} |A|^2\, \mathrm{d}x'\right\} = 0 \tag{2.7}$$

while Kodama and Wabnitz (1994) used averaged Lagrangian techniques to analyse pulses governed by the 'sliding-frequency' equation

$$u_z - \mathrm{i}\tfrac{1}{2}u_{xx} - \mathrm{i}|u|^2 u = \delta u + \beta(\partial_x - \mathrm{i}\alpha_0 z)^2 u. \tag{2.8}$$

This governs optical solitons in sub-oceanic transmission lines having amplifiers tuned to incrementally shifted central frequencies, so as to overcome timing 'jitter' otherwise introduced by repeated amplification of noise (Mollenauer *et al* 1992).

In (2.7) the integral term is an approximation (based on a limit of the Maxwell–Bloch equations) for the electric polarization pumped into the carrier frequency signal from a background medium containing two-level atoms. Since equation (2.7) yields the statement

$$-\mathrm{i}(A^* A_z + A A_z^*) + (A^* A_x - A A_x^*)_x + 2\mathrm{i}AA^*\left\{1 - \int_{-\infty}^{x} AA^*\, \mathrm{d}x'\right\} = 0$$

where * denotes a complex conjugate, Vanin *et al* (1994) readily deduced that for any pulse having $|A| \to 0$ as $x \to \pm\infty$ the 'pulse energy' $W \equiv \int_{-\infty}^{\infty} AA^* \, dx$ satisfies the evolution equation

$$\frac{dW}{dz} = 2W - W^2.$$

Consequently, a pulse will evolve towards a pulse with energy $W = 2$.

When *any* solution $A(x, z)$ to (2.7) is split into its modulus and argument as $A = F e^{i\theta}$, with F and θ real, the resulting equations are

$$F_{xx} + \left\{\theta_z - (\theta_x)^2\right\} F + F^3 = 0 \tag{2.9}$$

$$-F_z + 2\theta_x F_x + \theta_{xx} F + (1 - C)F = 0 \tag{2.10}$$

$$C_x = F^2 \qquad C(-\infty, z) = 0 \qquad C(\infty, z) = W(z) \tag{2.11}$$

which clearly admit travelling wave solutions with F, θ_x, θ_z and C depending only on $x - 2Vz$. Vanin *et al* (1994) used physical reasoning to argue that these solutions are unstable but that 'dissipative solitons' of the form

$$A = e^{i\theta(z,\eta)} F(\eta) \qquad C = C(\eta) \qquad \eta = x - \kappa z^2 \tag{2.12}$$

exist. Indeed, solutions with $\eta = x - 2Vz - \kappa z^2$ as in (2.2) exist.

Substituting $A = e^{i\theta(z,\eta)} F(\eta)$, $C = C(\eta)$ into (2.7), taking the imaginary part and then integrating and using the conditions $F \to 0$ as $\eta \to \pm\infty$ gives

$$2F^2(\theta_\eta + V + \kappa z) = C^2 - 2C.$$

This then yields the expression for the phase in the form (cf equation (2.5))

$$\theta = -(V + \kappa z)\eta + z\left(\Gamma - V^2 - \kappa V - \tfrac{1}{3}\kappa^2 z^2\right) + \Theta(\eta) \tag{2.13}$$

together with the system of ODEs

$$\begin{aligned} &F''(\eta) + \{\Gamma - \kappa\eta - (\Theta')^2\}F + F^3 = 0 \\ &F^2\Theta'(\eta) = \tfrac{1}{2}C^2 - C \qquad C'(\eta) = F^2 \end{aligned} \tag{2.14}$$

and the asymptotic conditions

$$C(-\infty) = 0 \qquad C(\infty) = 2 \qquad F \to 0 \quad \text{as} \quad \eta \to \pm\infty.$$

It should be noted that V does not arise in the system (2.14) and that, for $\kappa \neq 0$, the parameter Γ corresponds only to a shift in the origin for η. Thus, solutions describing accelerating pulses are those given by Vanin *et al* (1994), with κ playing the role of an eigenvalue related to the peak amplitude and to the asymmetry of the 'dissipative soliton'.

For equation (2.8), pulses of the form $u = e^{i\theta(z,\eta)} F(\eta)$, with $\eta = x - \frac{1}{2}\alpha_0 z^2 + bz$, are determined by Parker and Radha (2002), using a combination of asymptotic and numerical methods.

3. Photorefractive self-guiding beams

Equations of the form (2.1) arise in many treatments of 'spatial solitary waves', which are self-guiding beams of light. In particular, they describe beams propagating through photorefractive crystals in which the optical intensity causes an accumulation of a space charge and an associated transverse electric field. In Singh *et al* (1995), the equations for the envelopes

$u(x, z)$ and $v(x, z)$ of two orthogonal components of electric field at a common carrier frequency are given as

$$\begin{aligned} \mathrm{i}\alpha u_z + u_{xx} - \beta_1 u\{1 + |u|^2 + |v|^2\}^{-1} &= 0 \\ \mathrm{i}v_z + v_{xx} - \beta_2 v\{1 + |u|^2 + |v|^2\}^{-1} &= 0 \end{aligned} \tag{3.1}$$

where z is the propagation distance and x is a scaled distance transverse to the beam. While for small intensities this system (with $\alpha = 1$) has (3.1) as a limit (with $\sigma = 1$), unlike (1.1) it exhibits saturation of the nonlinearity. Christodoulides *et al* (1996) show that, when $\alpha = 1$, this system has both bright–bright and bright–dark solitary waves having $x - 2Vz$ as similarity variable and that each is stable for some ranges of parameters β_1 and β_2.

The system (3.1) omits effects due to the transverse diffusion of space charge. A model equation including such effects is

$$\mathrm{i}u_z + u_{xx} - \beta \frac{u}{1 + |u|^2} + \gamma \frac{\left(|u|^2_x\right) u}{1 + |u|^2} = 0. \tag{3.2}$$

Carvalho *et al* (1995) showed, by direct integration with initial data $u(0, x) = y(x)$ where $y(x)\mathrm{e}^{\mathrm{i}\mu z}$ satisfies the diffusionless limit ($\gamma = 0$) of (3.2), that a localized beam propagates confined to the immediate vicinity of a parabolic path $x \propto z^2$. However, there is no mention in the cited references that (3.2), like its generalization

$$\begin{aligned} \mathrm{i}\alpha u_z + u_{xx} - \frac{\beta_1 - \left(\gamma_1 |u|^2 + \delta_1 |v|^2\right)_x}{1 + |u|^2 + |v|^2} u &= 0 \\ \mathrm{i}v_z + v_{xx} - \frac{\beta_2 - \left(\delta_2 |u|^2 + \gamma_2 |v|^2\right)_x}{1 + |u|^2 + |v|^2} v &= 0 \end{aligned} \tag{3.3}$$

to include two polarizations as in (3.1), possesses exact ‘accelerating’ similarity solutions describing self-similar photorefractive beams which follow parabolic paths.

Using the same composite similarity variable η as for the systems (2.1) and (2.9)–(2.11) shows that a possible form of solutions to the system (3.3) is

$$u = \mathrm{e}^{\mathrm{i}\theta(z,\eta)} F(\eta) \qquad v = \mathrm{e}^{\mathrm{i}\phi(z,\eta)} G(\eta) \qquad \eta \equiv x - 2Vz - \kappa z^2.$$

Expressions for u_z, u_{xx}, v_z and v_{xx} are similar to those arising in (2.3). Inserting these together with $(|u|^2)_x = 2FF'(\eta)$ and $(|v|^2)_x = 2GG'(\eta)$ into (3.3) and investigating the imaginary part, yields expressions similar in form to (2.4) and (2.5); namely

$$\begin{aligned} \theta(z, \eta) &= \alpha\{\Phi(z, \eta) + \Gamma_1 z\} + \int C_1 F^{-2}\, \mathrm{d}\eta \\ \phi(z, \eta) &= \Phi(z, \eta) + \Gamma_2 z + \int C_2 G^{-2}\, \mathrm{d}\eta \\ \Phi(z, \eta) &\equiv (V + \kappa z)\eta + \tfrac{1}{3}\kappa^2 z^3 + \kappa V z^2. \end{aligned} \tag{3.4}$$

The corresponding ODEs describing the structure of the beam envelopes are

$$\begin{aligned} &F''(\eta) + \alpha^2 \left(V^2 - \kappa\eta - \Gamma_1\right) F - C_1^2 F^{-3}(\eta) - \frac{\beta_1 - 2\gamma_1 FF' - 2\delta_1 GG'}{1 + F^2 + G^2} F = 0 \\ &G''(\eta) + \left(V^2 - \kappa\eta - \Gamma_2\right) G - C_2^2 G^{-3}(\eta) - \frac{\beta_2 - 2\delta_2 FF' - 2\gamma_2 GG'}{1 + F^2 + G^2} G = 0. \end{aligned} \tag{3.5}$$

It is to be expected that (3.5) possesses isolated beam solutions only for certain combinations of the parameters κ, $V^2 - \Gamma_i$, α, β_i, γ_i and δ_i ($i = 1, 2$). Like (2.6), it becomes autonomous for $\kappa = 0$ (cf travelling wave pulses). For $\kappa \neq 0$, there is no loss of generality in setting $V = 0$, since change in V merely shifts the origin of z in the definition of η and only $V^2 - \Gamma_1$, $V^2 - \Gamma_2$ enter (3.5).

For isolated beams (or pulses) satisfying (3.3), the parameters C_1 and C_2 must be zero, so that the integrals in (3.4) are replaced by constants. In this case, the parameters κ and $\Gamma_2 - \Gamma_1$ are determined as part of a nonlinear eigenvalue problem, within which a solution having algebraic Airy function asymptotics (as $\kappa\eta \to -\infty$) is connected to exponentially decaying behaviour as $\kappa\eta \to \infty$.

4. The localized beam

This paper confines attention to the simpler eigenvalue problem arising from solutions of the form $u = \mathrm{e}^{\mathrm{i}\theta(z,\eta)}F(\eta)$, $\eta = x - 2Vz - \kappa z^2$ to the single complex equation (3.2). This gives

$$\theta(z,\eta) = (V + \kappa z)\eta + \tfrac{1}{3}\kappa^2 z^3 + \kappa V z^2 + \Gamma z$$

as in (3.4), and determines the profile of a localized beam as a solution to the ODE

$$F''(\eta) + (V^2 - \Gamma - \kappa\eta)F - \frac{(\beta - 2\gamma F F')F}{1 + F^2} = 0 \tag{4.1}$$

for which $F \to 0$ as $\eta \to \pm\infty$. For specified β and γ, this 'connection problem' is found to relate the beam 'curvature' parameter κ to the power within the beam.

In analysing (4.1), a rescaling

$$\eta = \beta^{-1/2}\zeta \qquad \gamma = \tfrac{1}{2}\beta^{1/2}\delta \qquad a \equiv -\beta^{-3/2}\kappa \qquad V^2 - \Gamma - \kappa\eta \equiv \beta M(\zeta)$$

so that $M(\zeta) = a[\zeta + (V^2 - \Gamma)/(a\beta)]$ helps. This replaces equation (4.1) by

$$F''(\zeta) + (M(\zeta) - 1)F = \frac{-F^2(F + \delta F')}{1 + F^2} \tag{4.2}$$

where the right-hand side is $\mathrm{O}(|F|^3)$ and $a = M'(\zeta)$ is a scaled beam curvature. In the special case $a = 0$, $\delta = 0$ (i.e. $\kappa = 0$ and $\gamma = 0$), equation (4.2) is autonomous and conservative, with first integral

$$[F'(\zeta)]^2 + MF^2 - \ln(1 + F^2) = c_0 \qquad \text{where} \quad M = \text{constant}.$$

The special case $c_0 = 0$ corresponds to isolated beams (in which $F \to 0$, $F' \to 0$ simultaneously). In the F, F' phase plane, it describes homoclinic orbits with $\mu \equiv \max|F|$ related to M through

$$M = \mu^{-2}\ln(1 + \mu^2) \equiv \tilde{M}(\mu). \tag{4.3}$$

This shows that these self-similar photorefractive beams can exist only for $0 < M < 1$. For each $\mu > 0$, there exist corresponding beam profiles which are symmetric about an *arbitrary* location $\zeta = \zeta^*$. They may be written as $F(\zeta) = G(\zeta - \zeta^*; \mu)$, where the *even* function $G(\xi; \mu)$ is defined through

$$[G_\xi(\xi;\mu)]^2 = \ln(1 + G^2) - \tilde{M}(\mu)G^2 \qquad G(0;\mu) = \mu \qquad G_\xi(0;\mu) = 0 \tag{4.4}$$

so that

$$G_{\xi\xi}(\xi;\mu) + \tilde{M}(\mu)G = \frac{G}{1 + G^2} \qquad \xi = \zeta - \zeta^*. \tag{4.5}$$

In this (non-diffusive) limit, beams travel along straight paths ($\kappa = 0$), while $\Gamma = V^2 - \beta\tilde{M}(\mu)$.

In the diffusive case ($\gamma > 0$), beams must be curved. This follows from the identity

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\{[F'(\zeta)]^2 + M(\zeta)F^2 - \ln(1 + F^2)\} = -2\delta\frac{F^2(F')^2}{1 + F^2} + aF^2$$

which follows from (4.2) since $M'(\zeta) = a$. Because the term in braces vanishes as both $\zeta \to -\infty$ and $\zeta \to \infty$, integration over $(-\infty, \infty)$ yields

$$a \int_{-\infty}^{\infty} F^2(\zeta)\,\mathrm{d}\zeta - 2\delta \int_{-\infty}^{\infty} \frac{F^2[F'(\zeta)]^2}{1+F^2}\,\mathrm{d}\zeta = 0. \tag{4.6}$$

Since both integrals are non-negative and since $\delta > 0$, it follows that profiles of isolated beams can exist only if $a > 0$. The corresponding self-similar beam will travel along a parabolic path, with 'curvature' $\kappa = -\beta^{3/2}a$.

In beam profiles satisfying (4.2), the peak amplitude μ and the beam centre $\zeta = \zeta^*$ at which $F'(\zeta^*) = 0$ and $F(\zeta^*) = \mu$ are both related to a and δ through (4.6) and through

$$\mu^2\{M(\zeta^*) - \tilde{M}(\mu)\} = a \int_{-\infty}^{\zeta^*} F^2(\zeta)\,\mathrm{d}\zeta - 2\delta \int_{-\infty}^{\zeta^*} \frac{F^2[F'(\zeta)]^2}{1+F^2}\,\mathrm{d}\zeta \tag{4.7}$$

which is obtained by integration over $(-\infty, \zeta^*)$. In direct numerical search for profiles, a further shift of origin, using $F(\zeta) = P(\psi)$ with $\psi = \zeta + (V^2 - \Gamma)/(a\beta)$, gives $M(\zeta) = a\psi$ and removes the parameters $V^2 - \Gamma$ and β from (4.2). The defining system becomes

$$\frac{\mathrm{d}^2P}{\mathrm{d}\psi^2} + (a\psi - 1)P + \frac{P^2(P + \delta P')}{1+P^2} = 0 \qquad \text{with} \quad \mu = P(\psi^*) \quad P'(\psi^*) = 0 \tag{4.8}$$

$$a \int_{-\infty}^{\infty} P^2\,\mathrm{d}\psi - 2\delta \int_{-\infty}^{\infty} \frac{P^2(P'(\psi))^2}{1+P^2}\,\mathrm{d}\psi = 0 \tag{4.9}$$

$$\psi^* \equiv \zeta^* + (V^2 - \Gamma)/(a\beta) = \frac{\tilde{M}(\mu)}{a} + \frac{1}{\mu^2}\int_{-\infty}^{\psi^*} P^2\,\mathrm{d}\psi - \frac{2\delta}{a\mu^2}\int_{-\infty}^{\psi^*} \frac{P^2(P'(\psi))^2}{1+P^2}\,\mathrm{d}\psi. \tag{4.10}$$

Although, for chosen a and δ, the location of the beam profile along the ψ-axis is *a priori* unknown, it may be estimated for small δ using a perturbation method (see section 5). This then allows estimation of the location of the beam fringes $|\psi - \psi^*| = \mathrm{O}(\ln \varepsilon^{-1})$, where $F(\zeta) = P(\psi) = \mathrm{O}(\varepsilon)$ with $\varepsilon \ll 1$. In these regions, analysis of the linearization of equation (4.8) shows that in the fringe where $\psi > \psi^*$, a decaying solution with $|P| \to 0$ (in the 'stable manifold') must have $P \approx -(1 - a\psi)^{-1/2}P'$, while in the fringe lying in $\psi < \psi^*$ the corresponding ('unstable manifold') approximation is $P \approx (1 - a\psi)^{-1/2}P'$, provided that $a\psi \leqslant 1$.

This motivates the numerical solution procedure for the system (4.8)–(4.10). For chosen δ and a, a value $\bar{\psi} \leqslant a^{-1}$ is selected at which initial conditions

$$P(\bar{\psi}) = \varepsilon \qquad P'(\bar{\psi}) = -(1 - a\bar{\psi})^{1/2}\varepsilon$$

are applied (with e.g. $\varepsilon = 10^{-3}$). Equation (4.8) is then integrated for ψ decreasing (i.e. $\psi \leqslant \bar{\psi}$) until $P^2 + (P')^2$ first attains a minimum. Then $\bar{\psi}$ is adjusted so as to successively reduce this minimum value (a 'shooting method') to $\mathrm{O}(10^{-6})$ or smaller. For the resulting profiles, conditions (4.9) and (4.10) are used as checks. The location $\zeta = \zeta^*$ of the beam centre is then calculated as $\zeta^* = \psi^* - (V^2 - \Gamma)/(a\beta)$.

5. The perturbation method

Approximate solutions to (4.2) may be obtained based on the observation that, as $\delta \to 0$,

$$F(\zeta) \to G(\xi; \mu) \qquad \text{with} \qquad \xi \equiv \zeta - \zeta^*.$$

When $M(\zeta)$ is expressed as $M(\zeta) = a\xi + \alpha$ (with $\alpha = a\zeta^* + \beta^{-1}(V^2 - \Gamma) = M(\zeta^*)$), equation (4.2) may be rewritten as

$$\frac{\mathrm{d}^2F}{\mathrm{d}\xi^2} + (a\xi + \alpha - 1)F = \frac{-F^3}{1+F^2} - \delta\frac{F^2F'}{1+F^2}.$$

Solutions are again sought with *pulse centre* at $\xi = 0$, so that

$$\frac{\mathrm{d}F}{\mathrm{d}\xi} = 0 \quad \text{at} \quad \xi = 0 \qquad F \to 0 \quad \text{as} \quad \xi \to \pm\infty.$$

As $\delta \to 0$, it is found that $F \to G(\xi;\mu)$, $a \to 0$ and $\alpha \to \tilde{M}(\mu)$, as defined in (4.3). It is possible to expand in powers of δ, with free parameter μ chosen so that $F(\xi^*) = \mu$, as

$$\begin{aligned} F(\zeta) &= G(\xi;\mu) + \delta F_1(\xi;\mu) + \delta^2 F_2(\xi;\mu) + \cdots \\ a &= \delta a_1 + \delta^2 a_2 + \cdots \\ \alpha &= \tilde{M}(\mu) + \delta\alpha_1 + \delta^2\alpha_2 + \cdots. \end{aligned} \tag{5.1}$$

At $\mathrm{O}(\delta)$ this yields

$$F_1''(\xi;\mu) + \tilde{M}(\mu)F_1 + \frac{G^2-1}{(1+G^2)^2}F_1 = -(\alpha_1 + a_1\xi)G - \frac{G^2G'}{1+G^2} \tag{5.2}$$

where primes denote differentiation with respect to ξ and the conditions on $F(\zeta)$ require that

$$F_1(0;\mu) = 0 \qquad F_1'(0;\mu) = 0 \qquad F_1(\xi;\mu) \to 0 \quad \text{as} \quad \xi \to \pm\infty.$$

It may be observed, by differentiating the identity (4.5) partially with respect to ξ and μ in turn, that

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}G' + \tilde{M}(\mu)G' + \frac{G^2-1}{(1+G^2)^2}G' = 0$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}G_\mu + \tilde{M}(\mu)G_\mu + \frac{G^2-1}{(1+G^2)^2}G_\mu = -\tilde{M}'(\mu)G(\xi;\mu).$$

Hence the function $F_1(\xi;\mu)$ may be expressed as

$$F_1 = \alpha_1(\tilde{M}'(\mu))^{-1}G_\mu(\xi;\mu) + H_1(\xi;\mu) \tag{5.3}$$

where $G' \equiv \partial G/\partial\xi$, $G_\mu \equiv \partial G/\partial\mu$ and $H_1(\xi;\mu)$ is an *odd* function of ξ satisfying

$$\begin{aligned} &\frac{\mathrm{d}^2H_1}{\mathrm{d}\xi^2} + \tilde{M}(\mu)H_1 + \frac{G^2-1}{(1+G^2)^2}H_1 = -a_1\xi G(\xi;\mu) - \frac{G^2}{1+G^2}G'(\xi;\mu) \\ &H_1(0;\mu) = 0 \qquad H_1'(0;\mu) = 0 \qquad \text{with} \quad H_1(\xi;\mu) \to 0 \qquad \text{as} \qquad \xi \to \pm\infty. \end{aligned} \tag{5.4}$$

However, in (5.3) it is found that $\alpha_1 = 0$, from both the $\mathrm{O}(\delta)$ approximation to equation (4.7) and the condition $F_1(0) = 0$ (since $G_\mu(0;\mu) \neq 0$). Then, after rewriting (5.4) as

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\{G'H_1' - H_1G''\} = -a_1\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{2}\xi G^2\right) + \frac{1}{2}a_1G^2 - \frac{G^2(G')^2}{1+G^2}$$

it is found that as $\xi \to \pm\infty$ the decay conditions yield

$$a_1\int_0^\infty G^2(\xi;\mu)\,\mathrm{d}\xi = 2\int_0^\infty \frac{G^2(\xi;\mu)(G'(\xi;\mu))^2}{1+G^2(\xi;\mu)}\,\mathrm{d}\xi \tag{5.5}$$

(or an equivalent condition involving integrals from $-\infty$ to 0). Formula (5.5) for a_1 may alternatively be deduced by inserting the expansions (5.1) into the condition (4.7).

Using these results allows solutions to (4.2) to be written as

$$F(\zeta) = G(\xi;\mu) + \delta H_1(\xi;\mu) + \mathrm{O}(\delta^2) \qquad a = \delta a_1 + \mathrm{O}(\delta^2) \qquad \alpha = \tilde{M}(\mu) + \mathrm{O}(\delta^2) \tag{5.6}$$

where $G(\xi;\mu)$ is defined by (4.4) and $a_1(\mu)$ follows from (5.5). The odd function $H_1(\xi;\mu)$ defined by (5.4) then decays as $\xi \to \infty$. For the corresponding beam of peak intensity $F_{\max} = |u|_{\max} = \mu$, the centre travels along the parabola

$$x - 2Vz + 2\gamma\beta a_1(\mu)z^2 \approx \beta^{-1/2}\zeta^* \approx \frac{\tilde{M}(\mu)}{2\gamma a_1(\mu)} + \frac{\Gamma - V^2}{2\beta\gamma a_1(\mu)}$$

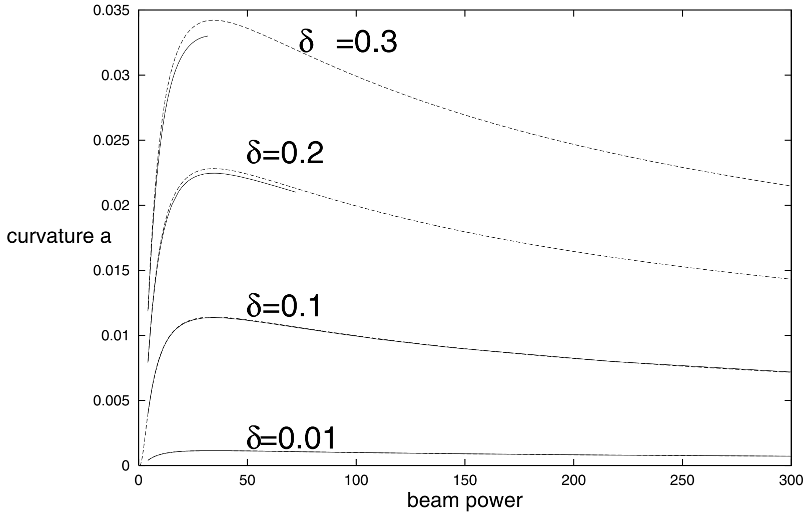

Figure 1. The variation in path curvature a with beam power for different values of δ; solid curves represent the numerical procedure, dashed curves represent the perturbation method.

and the phase is

$$\theta = \Gamma z + (V + \kappa z)(x - 2Vz) - \tfrac{2}{3}\kappa^2 z^3.$$

In these expressions μ, Γ and V are adjustable, while δ is $\delta \equiv 2\gamma\beta^{-1/2}$. The special choice $V = 0$, $\Gamma = -\beta\tilde{M}(\mu)$ leads to pulses with centre on the path $x = \kappa z^2 \approx -2\gamma\beta a_1(\mu) z^2$ and with phase $\theta = (\Gamma + \kappa x)z - \frac{2}{3}\kappa^2 z^3$.

6. Results and conclusions

In section 5, for specified δ and μ the predictions of $\alpha \approx \tilde{M}(\mu)$ and the path curvature $a \approx a_1\delta$ lead to $\psi^* \approx \tilde{M}(\mu)/(a_1\delta)$. Then, with $\xi = \zeta - \zeta^*$ the representation $F(\zeta) \approx G(\xi;\mu) + \delta F_1(\xi;\mu)$ agrees well, for $\delta = 0.01, 0.1$, with the numerically determined solution $F(\zeta) = P(\xi + \alpha/a)$ in section 4, where δ and a are the parameters specified. For $\delta = 0.2, 0.3$, the agreement is not so good (see figure 1), but the perturbation method still gives estimates of a as a function of δ and μ which are useful for the direct numerical integration.

In practice, dependence on *beam power*

$$\beta^{1/2} P \equiv \beta^{1/2} \int_{-\infty}^{\infty} F^2 \, \mathrm{d}\eta = \int_{-\infty}^{\infty} F^2 \, \mathrm{d}\zeta$$

is more useful than dependence on μ. Figure 1 shows that, for small δ, the scaled curvature a first increases with beam power, then decreases. Hence, for some values of a, two solutions for μ and ψ^* are to be expected from numerical search. This is confirmed in figures 2–4, where the solid curves correspond to higher beam powers, with larger values of μ and smaller values of ψ^*, while the dashed curves correspond to lower beam powers (with smaller μ and larger ψ^*). These figures also show that the dotted curves describing the perturbation theory approximation to the relation between the peak amplitude μ and the peak location $\psi = \psi^*$, and given by $\psi = \tilde{M}(\mu)/(a_1(\mu)\delta)$, are remarkably accurate. For larger diffusivity $\delta = 0.3$, it was possible to find beam profiles only for the lower beam powers (see figure 5), though the prediction $\psi^* \approx \tilde{M}(\mu)/(a_1(\mu)\delta)$ for the pulse centre remains good at these powers. In all cases, while the beam profile is necessarily asymmetric, the departure from symmetry is always found to be small. It is also confirmed that the beam power $\beta^{1/2}P(\mu)$ increases

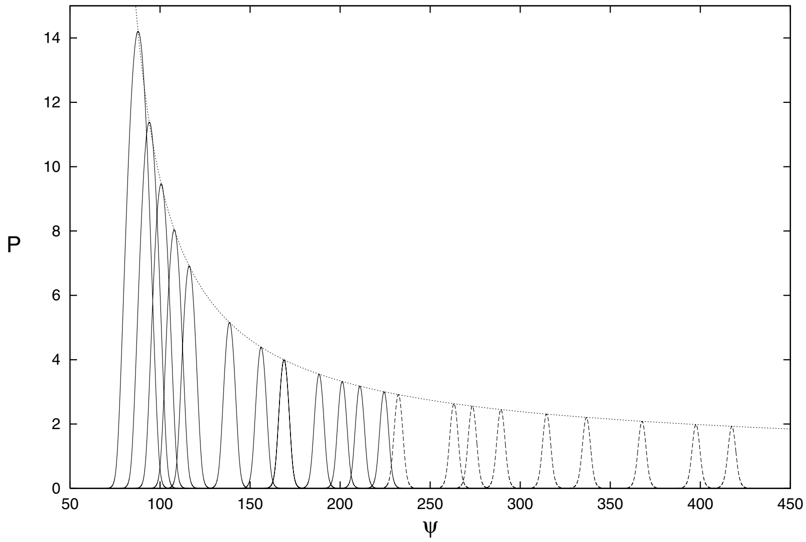

Figure 2. The beam profile $P(\psi)$ for $\delta = 0.01$ and for the various curvatures $a = 0.0003$, 0.0004, 0.0005, 0.0006, 0.0007, 0.0008, 0.0009, 0.001, 0.00105, 0.0011, 0.00112, 0.00113, 0.00114 corresponding to higher beam powers (solid curves from left to right) and for $a = 0.00114$, 0.001135, 0.00113, 0.00112, 0.0011, 0.00108, 0.00105, 0.00101, 0.001 corresponding to lower beam powers (dashed curves from left to right). The dotted curve represents the perturbation approximation $\psi^* = \tilde{M}(\mu)/(a_1(\mu)\delta)$ relating the maximum μ to the beam centre location ψ^*.

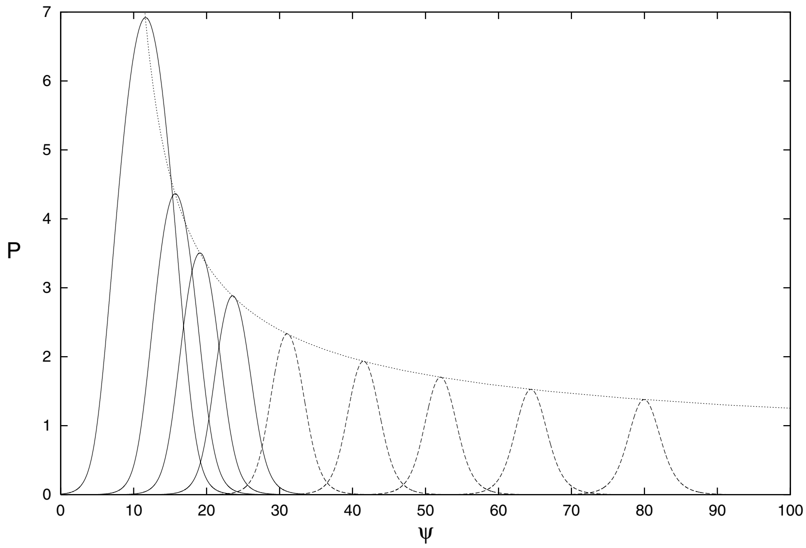

Figure 3. The beam profile $P(\psi)$ for $\delta = 0.1$ and for the various curvatures $a = 0.007$, 0.01, 0.011, 0.01136 corresponding to higher beam powers (solid curves from left to right) and for $a = 0.011$, 0.01, 0.009, 0.008, 0.007 corresponding to lower beam powers (dashed curves from left to right). The perturbation approximation $\psi^* = \tilde{M}(\mu)/(a_1(\mu)\delta)$ is again shown as a dotted curve.

monotonically with μ, while ψ^* decreases monotonically. Moreover, at fixed μ, $\beta^{1/2}P(\mu)$ increases only by less than 2% as δ increases from 0.01 to 0.3.

Direct numerical solution of equation (3.2) (Carvalho *et al* 1995) predicted parabolically curved beams. Hence, stability, at least for some μ and δ, is to be inferred. A fuller analysis of

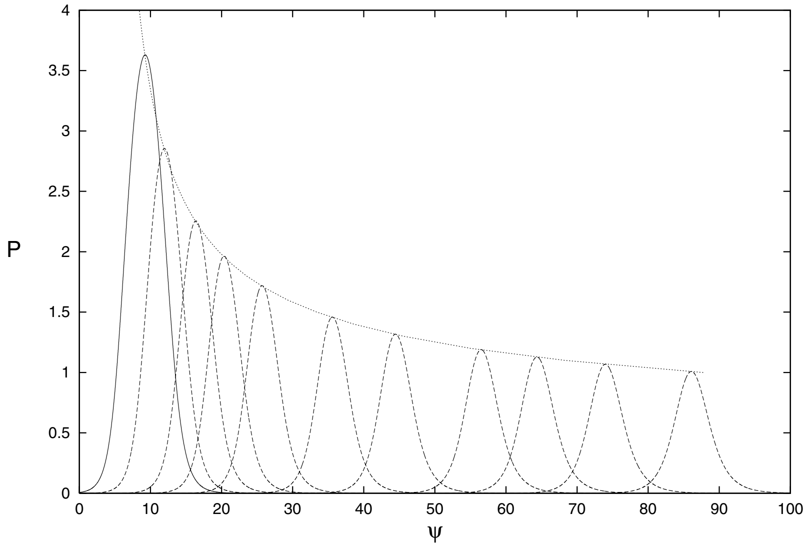

Figure 4. The beam profile $P(\psi)$ for $\delta = 0.2$ and for the curvature $a = 0.0215$ corresponding to a higher beam power (solid curve) and for $a = 0.02246$, 0.0215, 0.02, 0.018, 0.015, 0.013, 0.011, 0.01, 0.009, 0.008 corresponding to lower beam powers (dashed curves from left to right). The dotted curve represents $\psi^* = \tilde{M}(\mu)/(a_1(\mu)\delta)$.

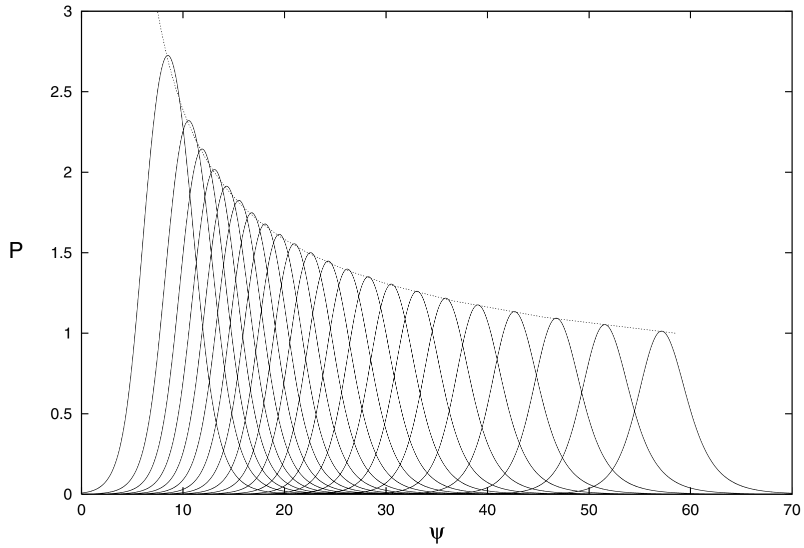

Figure 5. The beam profile $P(\psi)$ for $\delta = 0.3$ and for the various curvatures $a = 0.012$ to 0.033 in steps of 0.001 corresponding to lower beam powers (from right to left). The curve $\psi^* = \tilde{M}(\mu)/(a_1(\mu)\delta)$ is again shown as dotted.

the stability is currently underway. While, for all μ, the beam asymmetry is found to be small, the deviation from the familiar sech^2 profile may be substantial, and is due to the saturable nature of the nonlinearity. In experimental situations, a self-similar beam having permanent envelope profile will arise when initial profiles (and phases) are chosen closely consistent with (4.2).

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Appendix. The role of the F^{-3} terms

Terms similar to $F^{-3}(\eta)$ occur widely in the ODEs for the structure of modulated pulses (see (1.5), (2.6), (3.4) and also Parker (1988) and Manganaro and Parker (1993)). Their role is clarified as an 'angular momentum' effect, by use of the Clarkson–Kruskal (1989) technique. Applying this to the system (3.2) yields

$$u(z,x) = \mathrm{e}^{\mathrm{i}\alpha(\Phi(z,\eta)+\Gamma_1 z)} f(\eta) \qquad v(z,x) = \mathrm{e}^{\mathrm{i}(\Phi(z,\eta)+\Gamma_2 z)} g(\eta)$$

with $\eta = x - 2Vz - \kappa z^2$, with $\Phi(z,\eta)$ determined as in (3.3) and with the *complex* envelopes $f(\eta)$, $g(\eta)$ governed by

$$\begin{aligned} &f''(\eta) + \alpha^2\left(V^2 - \kappa\eta - \Gamma_1\right) f - \frac{\beta_1 - \gamma_1(f^*f' + ff^{*\prime}) - \delta_1(g^*g' + gg^{*\prime})}{1 + ff^* + gg^*} f = 0 \\ &g''(\eta) + \left(V^2 - \kappa\eta - \Gamma_2\right) g - \frac{\beta_2 - \delta_2(f^*f' + ff^{*\prime}) - \gamma_2(g^*g' + gg^{*\prime})}{1 + ff^* + gg^*} g = 0. \end{aligned} \tag{A.1}$$

These are equivalent to (3.4) under the substitutions

$$f(\eta) = F(\eta)\mathrm{e}^{\mathrm{i}\tilde{\theta}(\eta)} \quad g(\eta) = G(\eta)\mathrm{e}^{\mathrm{i}\tilde{\phi}(\eta)} \quad \tilde{\theta}'(\eta) = C_1 F^{-2} \quad \tilde{\phi}'(\eta) = C_2 G^{-2}. \tag{A.2}$$

Moreover, the system (A.1) yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\eta}(f^*f' - ff^{*\prime}) &= 0 = 2\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\eta}(F^2\tilde{\theta}') \\ \frac{\mathrm{d}}{\mathrm{d}\eta}(g^*g' - gg^{*\prime}) &= 0 = 2\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\eta}(G^2\tilde{\phi}') \end{aligned}$$

hence showing that the quantities $F^2\tilde{\theta}'$ and $G^2\tilde{\phi}'$ appear as angular momenta and are conserved as a consequence of (A.1). It is insertion of expressions (A.2) into the terms $(\tilde{\theta}')^2F$ and $(\tilde{\phi}')^2G$ which yields the $F^{-3}(\eta)$ and $G^{-3}(\eta)$ terms in (3.4). Similarly, the F^{-3} and G^{-3} terms in (1.5) and (1.7) arise directly from the amplitude-dependent contributions to the phases θ and ϕ in (1.4) and (1.6), respectively. For isolated pulses (or spatial beams) these two terms must vanish.

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